

B₂ Diagrams and Bhabha Equation with Mass Matrix *

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Till now nearly all papers about the Bhabha equation treat equations with a mass term $m\mathbf{1}$. This is merely a matter of mathematical simplicity, because we will see in this work that important physical fields can only be described by a mass term $\neq m\mathbf{1}$. The second aim of this publication is to derive a constructive procedure for the calculation of the representations of the Bhabha equation and the tensor and spinor field equations, which are equivalent to them.

1. Invariant Wave Equations

Because a differential equation of any order is being equivalent to a system of first order differential equations the field equation of a free field without subsidiary conditions can be written in the form:

$$\left(\frac{1}{i}\beta^\mu\partial_\mu + M\right)\psi = 0. \quad (1)$$

The invariance of this equation under infinitesimal Lorentz transformations

$$\psi' = (1 + i\varepsilon S_{lm})\psi \quad (2)$$

yields [1]:

$$[S_{lm}, \beta_j] + \beta_k(g_{lj}\delta_m^k - g_{mj}\delta_l^k) = 0, \quad (3)$$

$$[S_{lm}, M] = 0 \quad (4)$$

(we use the convention: $g_{11} = g_{22} = g_{33} = -g_{00} = -1$, $g_{ik} = 0$ for $i \neq k$). The six S_{lm} fulfill

$$\begin{aligned} [S_{lm}, S_{kp}] = & +g_{mk}S_{lp} + g_{lp}S_{mk} \\ & -g_{lk}S_{mp} - g_{mp}S_{lk}, \\ k, l, m, p \in \{0, 1, 2, 3\}. \end{aligned} \quad (5)$$

The S_{lm} can always be chosen in block-diagonal form and then (4) tells us that because of Schur's lemma M is a diagonal matrix.

$$S_{lm} = \begin{pmatrix} S_{lm}^1 & & \\ & \ddots & \\ & & S_{lm}^k \end{pmatrix} \Rightarrow M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_k \end{pmatrix}.$$

In order to get the explicit form of the matrices S_{lm} and β_j in a simple manner, one has to make a further assumption. There are two ways to achieve the same result:

1) Put

$$\begin{aligned} \beta_j &= S_{4j}, \quad \beta^j = S_4^j, \quad g_{44} = -1, \\ g_{4j} &= g_{j4} = 0, \\ j &\in \{0, 1, 2, 3\}. \end{aligned}$$

Now (3) has the same structure as (5) and demanding

$$[S_{4l}, S_{4m}] = -g_{44}S_{lm} = S_{lm} \quad (6)$$

(3), (4), (6) can be written:

$$\begin{aligned} [S_{lm}, S_{kp}] &= g_{mk}S_{lp} + g_{lp}S_{mk} - g_{lk}S_{mp} \\ &\quad - g_{mp}S_{lk}, \\ k, l, m, p &\in \{0, 1, 2, 3, 4\}. \end{aligned}$$

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That is why the S_{lm} and the β_j together are the generators of the 5-dimensional Lorentz algebra or of the B_2 Lie algebra.

2) We demand, that the S_{lm} and the β_j are the generators of a (half) simple Lie algebra. From (5) we conclude

$$[iS_{12}, S_{30}] = 0 \quad (7)$$

and (3) tells us, that

$$\begin{aligned} \rightarrow &:= \frac{1}{\sqrt{2}} (-iS_{41} + S_{42}), \\ \leftarrow &:= \frac{1}{\sqrt{2}} (iS_{41} + S_{42}), \\ \uparrow &:= \frac{1}{\sqrt{2}} (S_{43} + S_{40}), \\ \downarrow &:= \frac{1}{\sqrt{2}} (S_{43} - S_{40}) \end{aligned} \quad (8)$$

are step operators to $H_1 = iS_{12}$ and $H_2 = S_{30}$. The arrows indicate the directions in the $H_1 - H_2$ plane, in which the operators shift. But then the commutator relations between these operators are fixed [2] and therefore between the β_j too, namely (6). (1) and (6) together are called Bhabha equation.

The remaining S_{lm} form step operators too:

$$\begin{aligned} \nearrow &= \frac{S_{23} + S_{20} + iS_{31} + iS_{01}}{2}, \\ \nwarrow &= \frac{S_{23} - S_{20} - iS_{31} + iS_{01}}{2}, \\ \swarrow &= \frac{S_{23} + S_{20} - iS_{31} - iS_{01}}{2}, \\ \searrow &= \frac{S_{23} - S_{20} + iS_{31} - iS_{01}}{2}. \end{aligned} \quad (9)$$

Together with

$$\begin{aligned} [\leftarrow, \rightarrow] &= H_1, & [\downarrow, \uparrow] &= H_2, \\ [\nearrow, \nwarrow] &= H_1 + H_2, & [\swarrow, \searrow] &= H_1 - H_2 \end{aligned}$$

all informations about the commutator relations are contained in the following diagram due to the fact that the commutator between step operators is zero or a step operator with the sum of the eigenvalues.

$\begin{array}{c} 2 \\ \diagdown \\ 1 \end{array}$	\longleftrightarrow	\updownarrow	\times
\longleftrightarrow	H	+	-
\updownarrow	-	H	+
\times	+	-	H

(10)

The numbers give the position of the operators in the commutator. From the diagram you get for example:

$$\searrow = [\rightarrow, \downarrow].$$

The theory of Lie algebras now tells us that for any nonnegative integers l_1, l_2 there is a integer d

$$d = (l_1 + 1)(l_2 + 1)(1 + \tfrac{1}{2}(l_1 + l_2)) \cdot (1 + \tfrac{1}{3}(2l_1 + l_2))$$

and a d -dimensional representation of the B_2 -algebra

$$\{\beta_j, S_{lm}, M\} \in C^{d \times d}.$$

We denote them by

$$D^d(l_1, l_2).$$

A different common denotation is

$$R_5(l_1 + \tfrac{1}{2}l_2, \tfrac{1}{2}l_2).$$

The following table gives the dimensions of the lowest representations.

In the case l_2 odd, the representation $D(l_1, l_2)$ contains coupled spinor fields, and in the case l_2 even tensor fields. The symmetry type of the tensors follows from Weyl's branching rule [3]:

$$\begin{aligned} R_5(l_1 + \tfrac{1}{2}l_2, \tfrac{1}{2}l_2) &\rightarrow R_4(l_1 + \tfrac{1}{2}l_2, \tfrac{1}{2}l_2) \\ &+ R_4(l_1 + \tfrac{1}{2}l_2, \tfrac{1}{2}l_2 - 1) + \dots \\ &+ R_4(l_1 + \tfrac{1}{2}l_2, 0 \text{ or } \tfrac{1}{2}) \\ &+ R_4(l_1 - 1 + \tfrac{1}{2}l_2, \tfrac{1}{2}l_2) \\ &+ R_4(l_1 - 1 + \tfrac{1}{2}l_2, \tfrac{1}{2}l_2 - 1) + \dots \\ &+ R_4(l_1 - 1 + \tfrac{1}{2}l_2, 0 \text{ or } \tfrac{1}{2}) \\ &\vdots \\ &\vdots \\ &\vdots \\ &+ R_4(\tfrac{1}{2}l_2, \tfrac{1}{2}l_2) + R_4(\tfrac{1}{2}l_2, \tfrac{1}{2}l_2 - 1) + \dots \\ &+ R_4(\tfrac{1}{2}l_2, 0 \text{ or } \tfrac{1}{2}). \end{aligned}$$

Table 1. Dimensions of the lowest representations.

$l_2 \backslash l_1$	0	1	2	3	4	5	6	7
0	1	4	10	20	35	56	84	120
1	5	16	35	64	105	160	231	320
2	14	40	81	140	220	324	455	616
3	30	80	154	256	390	560	770	1024
4	55	140	260	420	625	880	1190	1560
5	91	224	405	640	935	1296	1729	2240
6	140	336	595	924	1330	1820	2401	3080
7	204	480	836	1280	1820	2464	3220	4096

A irreducible representation of the 4-dimensional Lorentz group $R_4(L_1, L_2)$ is in the case L_2 even a tensor, whose symmetry type is given by the Young tableau [2]:

$$\begin{array}{c} \boxed{} \boxed{} \boxed{} \boxed{} \cdots \boxed{} \boxed{} \boxed{} \boxed{} \cdots \boxed{} \boxed{} \\ \boxed{} \boxed{} \boxed{} \boxed{} \cdots \boxed{} \boxed{} \end{array}$$

The number of squares in the first and second column of a tableau cannot exceed four and the total antisymmetric tensor

$$\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \end{array}$$

is equivalent to the scalar and

$$\begin{array}{c} \boxed{} \boxed{} \cdots \\ \boxed{} \\ \boxed{} \end{array}$$

is associated to $\boxed{} \boxed{} \cdots$. Therefore we may conclude that for each tensor we can find a representation of the Bhabha equation containing it.

The spinor case is much simpler as a irreducible spinor is symmetric in its dotted and undotted indices.

2. B_2 Diagrams

There are two fundamental representations of the B_2 algebra, namely $D^5(1, 0)$ and $D^4(0, 1)$, which are given by Fig. 1 and 2. We label the roots in the $H_1 - H_2$ plane by

$$\boxed{\begin{array}{c} H_1, H_2 \\ A \end{array}} \quad \text{or by} \quad |H_1, H_2, A\rangle,$$

where $A = (K_1, K_2)$ is invariant against the 4-dimensional Lorentz group.

$$\begin{aligned} -K_1 &= \rightarrow\leftarrow + \leftarrow\rightarrow + \uparrow\downarrow + \downarrow\uparrow \\ K_3 &= H_1 H_2 \\ &+ \frac{1}{2}(\nearrow\searrow + \searrow\swarrow - \nearrow\swarrow - \searrow\nwarrow). \end{aligned}$$

We use the convention $\bar{a} = -a$ for numbers (see Fig. 3, 4) and $\bar{A} = (K_1, -K_2)$.

The numbers beside the squares indicate how many times you get the neighbouring root after application of one of the shift operators (8). The oblique lines have the same meaning, but only numbers different from 1 are added to the oblique lines. The operators (8) are combinations of the β_j , while (9) are combinations of the 4-dimensional Lorentz transformations. That is why roots belonging to the same irreducible representation are connected by oblique lines. The roots are normalized so that the upwards oblique shifting operators give a positive multiple of the root in question (and the downwards shifting operators the negative multiple) and that for the operators (8) the back transformation gives the negative multiple.

With the aid of the commutator relations (10) it is easy to compute all numbers contained in such a diagram. In the case of $D^5(1, 0)$ for instance you have:

$$\begin{aligned} \rightarrow|1, 0, A\rangle &= 0 \quad \leftarrow|1, 0, A\rangle = a|0, 0, B\rangle \\ \rightarrow|0, 0, B\rangle &= -a|1, 0, A\rangle \\ \Rightarrow -a^2|1, 0, A\rangle \\ &= \rightarrow\leftarrow|1, 0, A\rangle = [\rightarrow, \leftarrow]|1, 0, A\rangle \\ &= -H_1|1, 0, A\rangle = -1|1, 0, A\rangle \\ \Rightarrow a &= \pm 1. \end{aligned}$$

In this manner we computed all B_2 -diagrams up to dimension 35, starting at the highest weigh of the representation $D^d(l_1, l_2)$, i.e. the root with the coordinate $(l_1 + \frac{1}{2}l_2, \frac{1}{2}l_2)$ [4].

3. B_2 Diagrams and Bhabha Equation

From the definition

$$\begin{aligned} d^\pm &= -\frac{1}{\sqrt{2}}(\pm \partial_1 - i \partial_2), \\ t^\pm &= \frac{1}{\sqrt{2}} \cdot \frac{1}{i}(-\partial_3 \pm \partial_0) \end{aligned} \quad (11)$$

we conclude

$$\frac{1}{i} \beta^\mu \partial_\mu = \rightarrow d^+ + \leftarrow d^- + \uparrow t^+ + \downarrow t^-. \quad (12)$$

This relation allows to connect the B₂-diagrams with the Bhabha equation (1). We read off from Fig. 1 the matrix form of the Bhabha equation $D^5(1,0)$:

$$\begin{pmatrix} m_4 & d^- & d^+ & t^- & -t^+ \\ -d^+ & m_1 & & & \\ -d^- & & m_1 & & \\ -t^+ & & & m_1 & \\ +t^- & & & & m_1 \end{pmatrix} \circ \begin{pmatrix} |0,0,B\rangle \\ |1,0,A\rangle \\ |\bar{1},0,A\rangle \\ |0,1,A\rangle \\ |0,\bar{1},A\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (13)$$

We define

$$\begin{aligned} \Psi_1 &= \frac{+|1,0\rangle - |\bar{1},0\rangle}{\sqrt{2}} \\ \Psi_2 &= -\frac{|1,0\rangle + |\bar{1},0\rangle}{i\sqrt{2}} \\ \Psi_3 &= -\frac{|0,1\rangle - |0,\bar{1}\rangle}{i\sqrt{2}} \\ \Psi_0 &= -\frac{|0,1\rangle + |0,\bar{1}\rangle}{i\sqrt{2}} \\ \Phi &= |0,0\rangle \end{aligned} \quad (14)$$

and get a equivalent form of (13), namely the set of tensor equations

$$\begin{aligned} m_4 \Phi + \partial_j \Psi^j &= 0, \\ m_1 \Psi_j - \partial_j \Phi &= 0. \end{aligned} \quad (15)$$

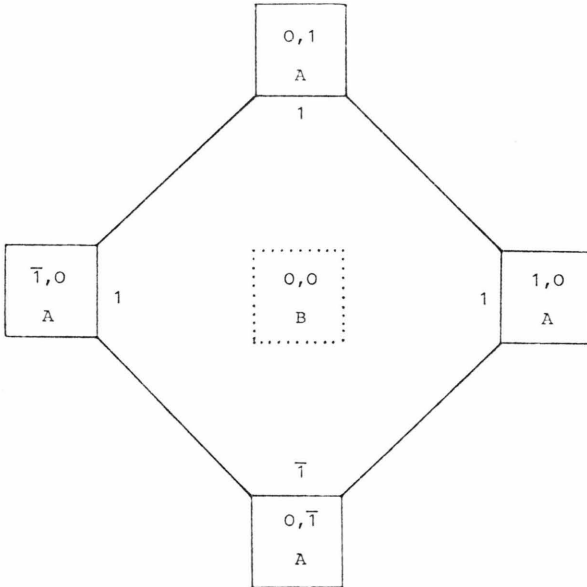


Fig. 1. $D^5(1,0)$ $A = (1,0)$, $B = (4,0)$.

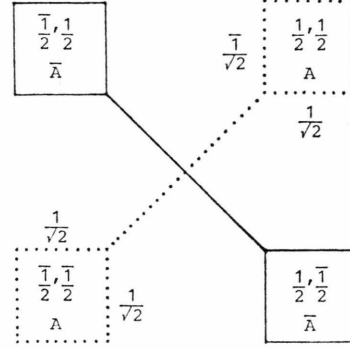


Fig. 2. $D^4(0,1)$ $A = (1, \frac{3}{4})$, $\bar{A} = (1, \frac{3}{4}\text{-bar})$.

This is the well known spin 0 DKP equation.

From Fig. 2 we don't get tensor but spinor equations. With the aid of the Pauli matrices we compute the spinor form of the gradient:

$$\begin{aligned} \partial^{1\dot{1}} &= \partial_0 + \partial_3 = -i\sqrt{2}t^- = \partial_{2\dot{2}}, \\ \partial^{2\dot{2}} &= \partial_0 - \partial_3 = i\sqrt{2}t^+ = \partial_{1\dot{1}}, \\ \partial^{1\dot{2}} &= \partial_1 + i\partial_2 = \sqrt{2}d^- = -\partial_{2\dot{1}}, \\ \partial^{2\dot{1}} &= \partial_1 - i\partial_2 = -\sqrt{2}d^+ = -\partial_{1\dot{2}}. \end{aligned} \quad (16)$$

Now we define

$$\begin{aligned} \varphi_1 &= \varphi^2 = |\tfrac{1}{2}, \tfrac{1}{2}, A\rangle, \quad \varphi_2 = -\varphi^1 = i|\tfrac{1}{2}, \tfrac{1}{2}, \bar{A}\rangle, \\ \chi^1 &= \chi^{\dot{2}} = |\tfrac{1}{2}, \tfrac{1}{2}, \bar{A}\rangle, \quad \chi^{\dot{2}} = -\chi^1 = -i|\tfrac{1}{2}, \tfrac{1}{2}, A\rangle, \end{aligned}$$

and yield the spinor equations equivalent to Fig. 2:

$$\begin{aligned} m\varphi_\beta + \tfrac{1}{2}\partial_{\beta\dot{\alpha}}\chi^{\dot{\alpha}} &= 0, \\ \tilde{m}\chi_{\dot{\alpha}} + \tfrac{1}{2}\partial_{\beta\dot{\alpha}}\varphi^\beta &= 0. \end{aligned} \quad (17)$$

Now and further we demand not only invariance against Lorentz rotations, but also invariance against parity-transformations. Concerning the B₂-diagrams this means $H_2 \Rightarrow -H_2$, $H_1 \Rightarrow H_1$. That is why we set $m = \tilde{m}$. Now (17) is the spinor form of the Dirac equation.

4. Determinants and the Klein-Gordon Divisor

Before computing more complicated representations, let us take a further look on Equation (1). If (1) is satisfied, we can get a scalar wave equation as (1) implies:

$$\det\left(\frac{1}{i}\beta^\mu\partial_\mu + M\right)\Psi = 0. \quad (18)$$

From the Lorentz-invariance of (1) there follows the invariance of

$$W := \det \left(\frac{1}{i} \beta^\mu \partial_\mu + M \right). \quad (19)$$

Therefore W contains derivations only in the form

$$\square = 2(d^+ d^- + t^+ t^-). \quad (20)$$

For practical computation it is allowed to set $d^+ = d^- = d$, $t^+ = t^- = 0$. That is why each row of a B₂-diagram gives a multiplicative factor to W . In the case $M = m \mathbf{1}$ Bhabha showed that with $s = l_1 + \frac{1}{2} l_2$

I) l_2 even

$$W = m^{a(0)} \cdot (m^2 + \square)^{a(1)} \dots \cdot (m^2 + (s-1)^2 \square)^{a(s-1)} (m^2 + s^2 \square)^{a(s)};$$

II) l_2 odd

$$W = (m^2 + \frac{1}{4} \square)^{a(\frac{1}{2})} \dots (m^2 + s^2 \square)^{a(s)}. \quad (21)$$

$a(j)$ is the multiplicity of the eigenvalue j of β^0 [1]. In nearly all papers about invariant wave equations $a(j)$ is omitted. The computation of W for the fundamental representation is very easy and yields

$$W(1, 0) = m_1^3 (m_1 m_4 + \square),$$

$$W(0, 1) = (m^2 + \frac{1}{4} \square)^2.$$

$m_4 = 0$, $m_1 \neq 0$ describes a Klein-Gordon particle with zero rest mass and with $W(1, 0) = 0$. The importance of W relies on its connection with the Klein Gordon divisor d_{KG} , defined by (κ_j , not m_j are the masses of the fields!):

$$d_{\text{KG}} \cdot \left(\frac{1}{i} \beta^\mu \partial_\mu + M \right) = \prod_{j=1}^n (\square - \kappa_j^2) \mathbf{1}.$$

If d_{KG} exists, we know how to quantize the fields:

$$[\bar{\Psi}(x), \Psi(x)]_{\pm} \sim d_{\text{KG}} \Delta(x - x'),$$

where $\Delta(x - x')$ is a multimass invariant function [5]. We conclude from the definition of W :

$$W \neq 0 \Leftrightarrow \text{The Klein-Gordon divisor exists.}$$

As we can always take a representation in which M is a diagonal matrix and the β_j are nondiagonal, we may conclude from the definition of a determinant

$$\det(a_{ik}) = \sum_{\text{Per}} \sigma(\text{Per}) a_{1k_1} a_{2k_2} \dots a_{nk_n}$$

the following result:

$$\det M \neq 0 \Rightarrow \det \left(\frac{1}{i} \beta^\mu \partial_\mu + M \right) \neq 0$$

and together with (21) (The determinant is a continuous function of the m_j , in the case $m_j \Rightarrow m$ for all j we must get (21).):

$$l_2 \text{ odd} \Rightarrow \det \left(\frac{1}{i} \beta^\mu \partial_\mu + M \right) \neq 0$$

$$l_2 \text{ even}, M = 0 \Rightarrow \det \left(\frac{1}{i} \beta^\mu \partial_\mu + 0 \right) = 0.$$

Because d_{KG} contains at least one d'Alembert operator for each mass, we see that interaction terms with fields with high spin and therefore many masses cannot be renormalizable. But in the case $W = 0$, a renormalisation may be possible. An example of this fact gives the representation $D^{10}(0, 2)$ with a mass matrix, which we will consider now.

5. $D^{10}(0, 2)$

The B₂-diagram of this representation is shown in Figure 3. The tensor equations are

$$\begin{aligned} m_3 A^j + \partial_k F^{[kj]} &= 0, \\ m_2 F_{[kj]} - (\partial_k A_j - \partial_j A_k) &= 0, \end{aligned} \quad (22)$$

and

$$W(0, 2) = m_2^3 m_3 (m_2 m_3 + \square)^3.$$

Okubo and Tosa [6] pointed out, that after introducing a colour index and setting $m_3 = 0$, $m_2 = 1$

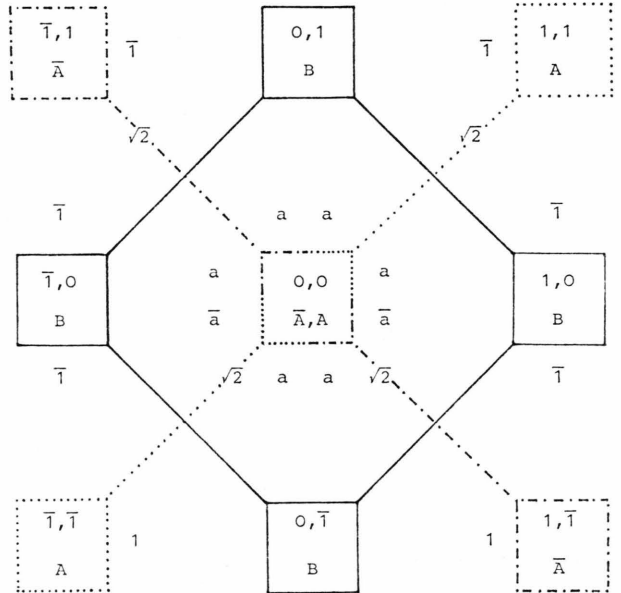


Fig. 3. $D^{10}(0,2)$ $A = (2,2)$, $\bar{A} = (2,\bar{2})$, $B = (3,0)$; $a = \frac{1}{\sqrt{2}}$.

the renormalizable Yang-Mills Lagrangian can be written in the form:

$$L = \bar{\Psi}_a \left(\frac{1}{i} \beta^\mu \partial_\mu + M \right) \Psi_a + \Gamma^{abc} \Psi_a \Psi_b \Psi_c.$$

Further we see, that in the case $m_3 = 0$, $m_2 = 1$, (22) describes the charge-free Maxwell equations.

6. The Spin $\frac{3}{2}$ Equations

There are two representations of spin $\frac{3}{2}$, $D^{16}(1, 1)$ and $D^{20}(0, 3)$. Here and in the following the (dotted or dashed) lines beside the numbers indicate which root they refer to.

From Fig. 4. we get

$$m_6 \varphi_\beta + \frac{3}{4} \partial_{\beta\alpha} \chi^\alpha + \sqrt{\frac{5}{8}} \partial^{\mu\nu} R_{\beta\mu\nu} = 0,$$

$$m_6 \chi_\alpha + \frac{3}{4} \partial_{\beta\alpha} \varphi^\beta + \sqrt{\frac{5}{8}} \partial^{\mu\nu} G_{\mu\nu\alpha} = 0,$$

$$m_2 R_{\alpha\beta\gamma} - \frac{1}{4} \sqrt{\frac{5}{2}} (\partial_{\beta\gamma} \varphi_\alpha + \partial_{\alpha\gamma} \varphi_\beta) - \frac{1}{2} (\partial_\alpha^\delta G_{\beta\gamma\delta} + \partial_\beta^\delta G_{\alpha\gamma\delta}) = 0,$$

$$m_2 G_{\alpha\beta\gamma} - \frac{1}{4} \sqrt{\frac{5}{2}} (\partial_{\alpha\beta} \chi_\gamma + \partial_{\alpha\gamma} \chi_\beta) - \frac{1}{2} (\partial_\beta^\delta R_{\delta\alpha\gamma} + \partial_\gamma^\delta R_{\delta\alpha\beta}) = 0,$$

$$W(1, 1) = (m_2^2 + \frac{1}{4} \square)^4 \left(\frac{9}{16} \square^2 + [30 m_2 m_6 + 9 m_2^2 + m_6^2] \frac{1}{16} \square + m_2^2 m_6^2 \right)^2.$$

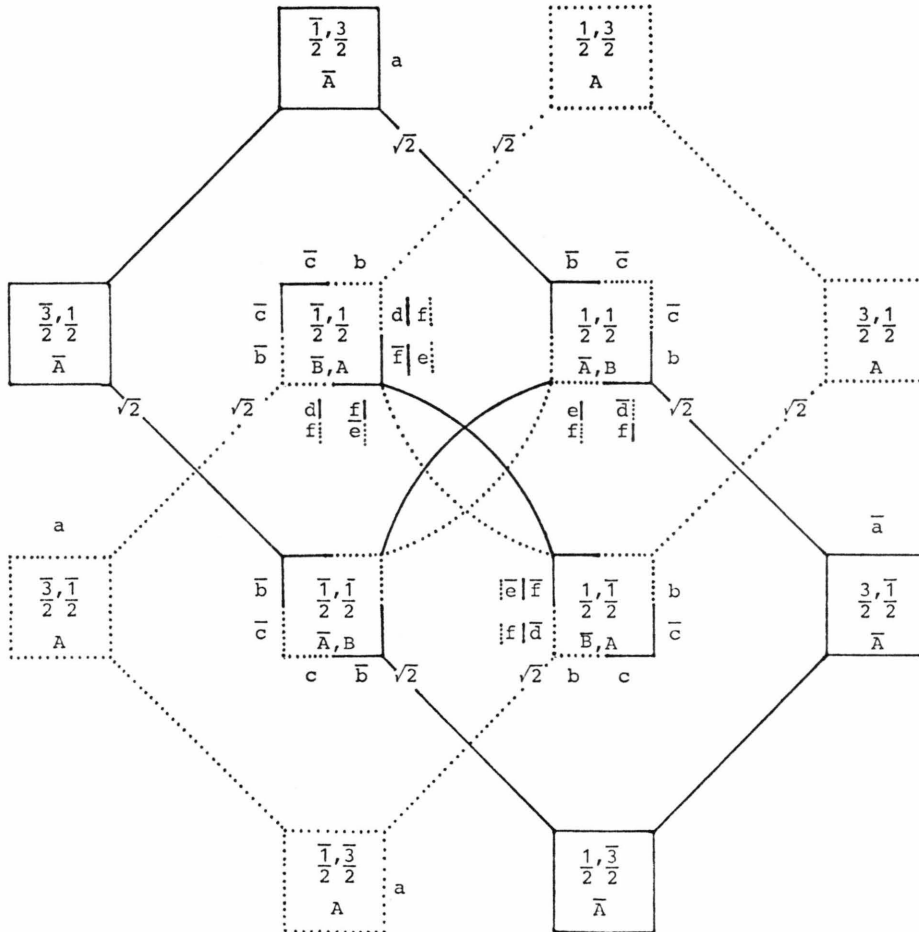


Fig. 4. $D^{16}(1,1)$ $A = (2, \frac{5}{4})$, $\bar{A} = (2, \frac{5}{4})$, $B = (6, \frac{3}{4})$, $\bar{B} = (6, \frac{3}{4})$; $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{2}$, $c = \frac{\sqrt{5}}{2}$,
 $d = \frac{1}{\sqrt{8}}$, $e = \frac{3}{\sqrt{8}}$, $f = \frac{\sqrt{5}}{\sqrt{8}}$.

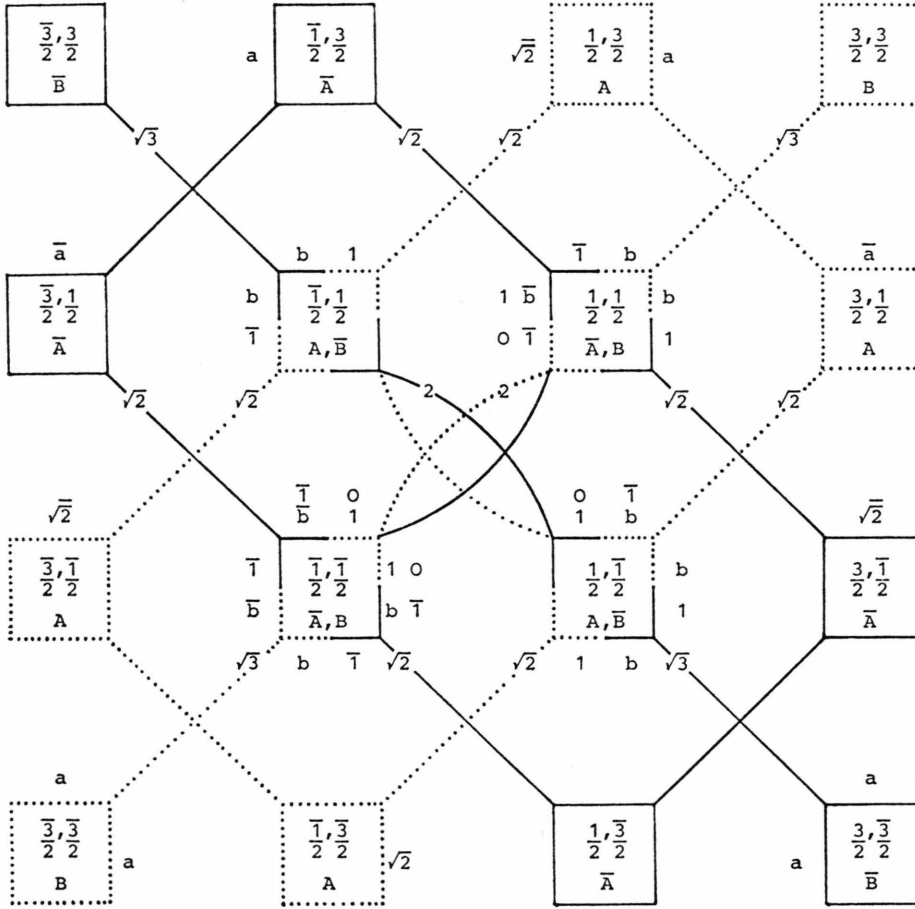


Fig. 5. $D^{20}(0,3)$ $A = \left(5, \frac{5}{4}\right)$, $\bar{A} = \left(5, \frac{5}{4}\right)$, $B = \left(3, \frac{15}{4}\right)$, $\bar{B} = \left(3, \frac{15}{4}\right)$; $a = \frac{\sqrt{3}}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$.

Figure 5 gives the equations

$$m_5 R_{\alpha\beta\gamma} - (\partial_\alpha^\delta G_{\beta\gamma\delta} + \partial_\beta^\delta G_{\alpha\gamma\delta}) + \frac{\sqrt{3}}{2} \partial_\gamma^\delta V_{\alpha\beta\delta} = 0,$$

$$\bar{m}_5 G_{\alpha\beta\gamma} - (\partial_\beta^\delta R_{\delta\alpha\gamma} + \partial_\gamma^\delta R_{\delta\alpha\beta}) + \frac{\sqrt{3}}{2} \partial_\alpha^\delta B_{\delta\beta\gamma} = 0,$$

$$m_3 V_{\alpha\beta\gamma} - \frac{1}{2\sqrt{3}} \cdot (\partial_\gamma^\delta R_{\alpha\beta\delta} + \partial_\alpha^\delta R_{\beta\gamma\delta} + \partial_\beta^\delta R_{\gamma\alpha\delta}) = 0,$$

$$\bar{m}_3 B_{\alpha\beta\gamma} - \frac{1}{2\sqrt{3}} \cdot (\partial_\alpha^\delta G_{\delta\beta\gamma} + \partial_\beta^\delta G_{\delta\gamma\alpha} + \partial_\gamma^\delta G_{\delta\alpha\beta}) = 0,$$

$$W(0,3) = (m_5^2 + \frac{1}{4}\square)^2 \cdot (\frac{9}{16}\square^2 + (\frac{3}{2}m_3m_5 + m_3^2)\square + m_3^2m_5^2)^4.$$

In the case $m_3 = -3m_5$ we get $W(0,3) = 9^4(m_5^2 + \frac{1}{4}\square)^{10}$ and therefore the fields describe a particle with one spin and one mass, which is impossible with a mass term $m1$!

7. The Spin 2 Equations

The simplest spin 2 representation is $D^{14}(2,0)$ with the diagram given by Fig. 6 and with the field equations:

$$m_{10}\varphi + \sqrt{\frac{5}{2}}\partial^j A_j = 0,$$

$$m_7 A_j - \sqrt{\frac{5}{2}}\partial_j \varphi + \partial^k h_{(kj)} = 0,$$

$$m_2 h_{kj} - (\partial_j A_k + \partial_k A_j) + \frac{1}{2}g_{kj}\partial^l A_l = 0,$$

$$W(2,0) = m_5^2 (m_2 m_7 + \square)^3 \cdot \left(m_2 m_7 m_{10} + \frac{5m_2 + 3m_{10}}{2} \square \right).$$

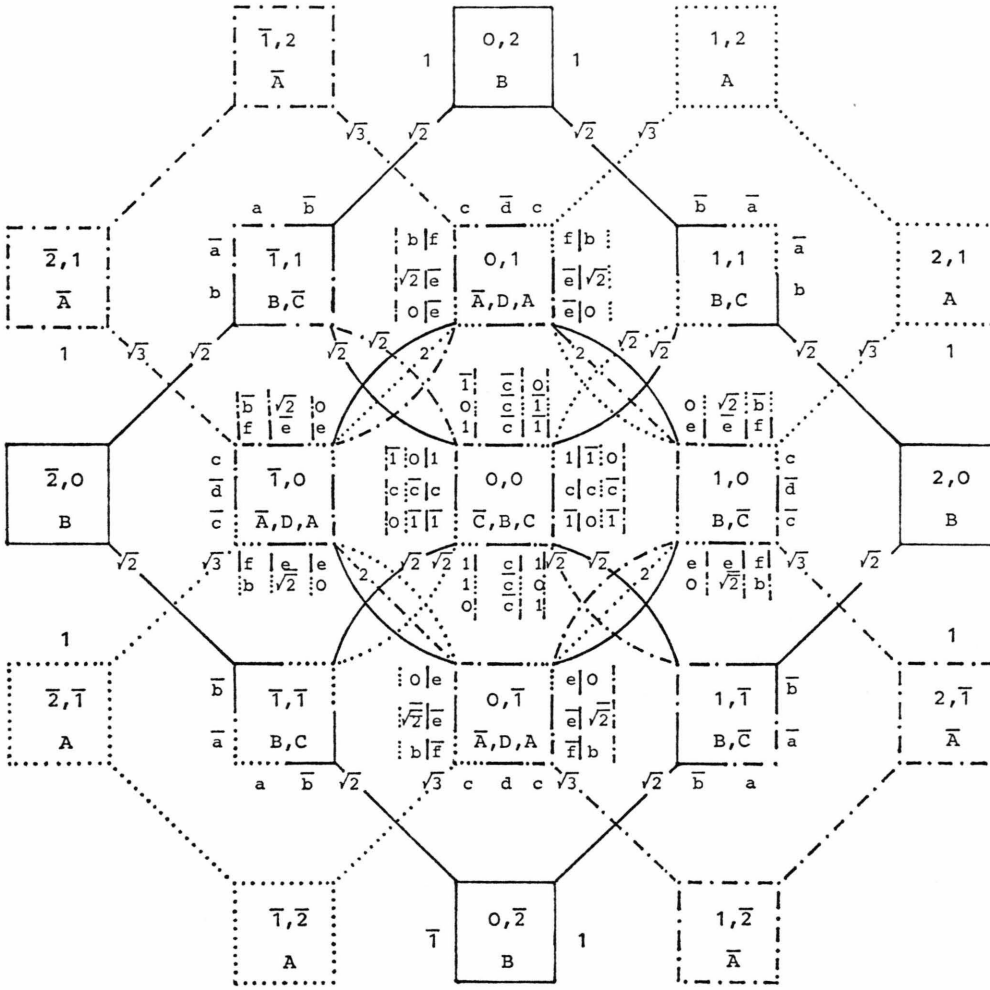


Fig. 7. $D^{35}(1,2)$ $A = (3,3)$, $\bar{A} = (3,\bar{3})$, $B = (4,0)$, $C = (8,2)$, $\bar{C} = (8,\bar{2})$, $D = (9,0)$; $a = \frac{\sqrt{3}}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$, $c = \frac{1}{\sqrt{3}}$,
 $d = \frac{2}{\sqrt{3}}$, $e = \frac{\sqrt{2}}{\sqrt{3}}$, $f = \frac{1}{\sqrt{6}}$.

From Fig. 8 we compute the tensor equations of $D^{35}(0,4)$:

$$\begin{aligned}
 m_8 H_{kl} - \partial^l (T_{jkl} + T_{klj}) &= 0, \\
 m_7 T_{kjl} + \frac{3}{\sqrt{2}} (\partial_j H_{kl} - \partial_l H_{kj}) - \frac{1}{\sqrt{2}} \partial^n (g_{kj} H_{nl} - g_{kl} H_{nj}) + \partial^n C_{nkjl} &= 0, \\
 m_4 C_{klmn} - \{ \partial_k T_{lmn} - \partial_l T_{knm} + \partial_m T_{nkl} - \partial_n T_{mkl} - \frac{1}{2} \partial^p (T_{lpn} + T_{npl}) g_{km} \\
 &\quad - (T_{lpm} + T_{mpl}) g_{kn} + (T_{kpm} + T_{mpk}) g_{ln} - (T_{kpn} + T_{npk}) g_{lm} \} = 0, \\
 W(0,4) &= m_8 m_7^3 (m_7 m_8 + \square)^3 (m_4 m_7 + \square)^5 (m_4 m_7 m_8 + (m_8 + 3 m_4) \square)^5.
 \end{aligned}$$

For all spin 2 representations there are special mass matrices, which allow for an interpretation in terms

of the linearized gravitation theory [7, 4]. It is not surprising that the symmetric tensor H_{jk} may be

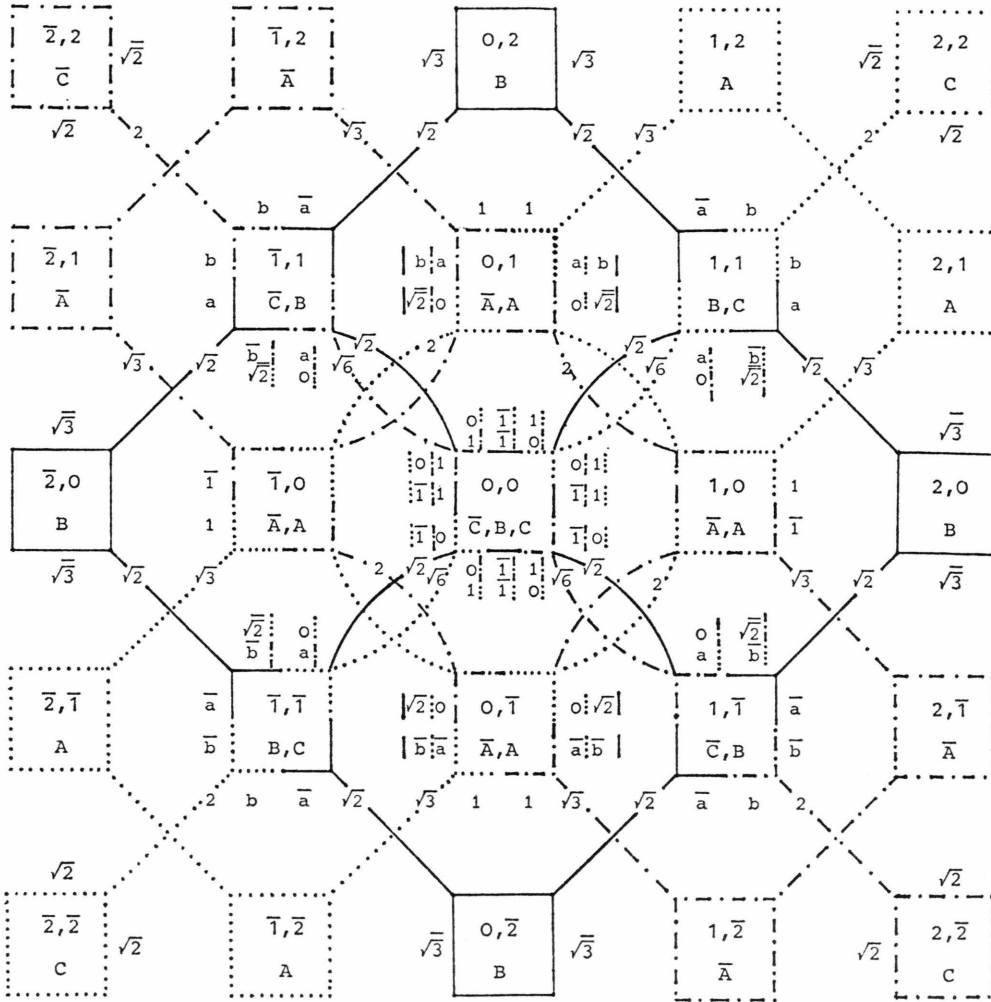


Fig. 8. $D^{35}(0,4)$ $A = (7,3)$, $\bar{A} = (7,\bar{3})$, $B = (8,0)$, $C = (4,6)$, $\bar{C} = (4,\bar{6})$; $a = \frac{\sqrt{3}}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$.

considered as weak gravitational field. $T_{k[jl]}$, which is antisymmetric in the last two indices, is connected with Freud's superpotential and $C_{[kl][mn]}$ with the Riemann tensor.

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